VII Introduction to Convex Surfaces

let (M, ξ) be a contact 3-manifold

a vector field v is a contact vector field if its flow preserves ξ

if α is a contact form for ξ, then v is a contact vector field

if flow ϕ_t of v preserves $\xi = \ker \alpha$

which is equivalent to

pos. function

$$\mathcal{L}_v \alpha = \frac{d}{dt} \phi_t^* \alpha \Big|_{t=0} = \frac{d}{dt} f_t \alpha \Big|_{t=0} = g\alpha$$

for any $g : M \to \mathbb{R}$

so $\boxed{v \text{ is a contact vector field} \iff \mathcal{L}_v \alpha = g\alpha}$

<u>example:</u>

X_α Reeb vector field of α

$$\mathcal{L}_{X_\alpha} \alpha = \iota_{X_\alpha} d\alpha + d\iota_{X_\alpha}\alpha = 0 + d1 = 0$$

$\therefore X_\alpha$ a contact vector field

note also X_α is transverse to ξ

<u>exercise:</u>

① show a contact vector field v is a Reeb field for some $\alpha \iff v$ is transverse to ξ

② show a contact vector field v is always tangent to ξ $\iff v = 0$

lemma 1:

(M, ξ) a contact manifold

α a contact form for ξ

a vector field v is a contact vector field

$$\Longleftrightarrow$$

there is a function $H: M \to \mathbb{R}$ s.t.

$$\alpha(v) = -H$$

Reeb field

$$\iota_v d\alpha = dH - (dH(X_\alpha))\alpha$$

Proof: assume v is a contact vector field

set $H = -\alpha(v)$

now $g\alpha = \mathcal{L}_v\alpha = d\iota_v\alpha + \iota_v d\alpha = -dH + \iota_v d\alpha$

so $\iota_v d\alpha = dH + g\alpha$

plug Reeb field X_α into equation to get

$$0 = dH(X_\alpha) + g$$

so $\iota_v d\alpha = dH - dH(X_\alpha)\alpha$

now if v satisfies equations then

$$\mathcal{L}_v\alpha = \iota_v d\alpha + d\iota_v\alpha = dH - (dH(X_\alpha))\alpha - dH$$
$$= -dH(X_\alpha)\alpha$$

so v contact field

exercise:

given $H: M \to \mathbb{R}$ there is some vector field v satisfying equations in lemma 1

Remark: this says any locally defined contact vector field can be extended to a global one a surface Σ in a contact manifold (M,ξ) is convex if there is a contact vector field v transverse to Σ

lemma 2:

a surface Σ is convex $\Leftrightarrow$ $\exists$ an embedding $\Sigma\times\mathbb{R}\xrightarrow{\phi} M$ such that $\phi(\Sigma\times\{0\})=\Sigma$ and $\phi^*(\xi)$ is vertically invariant (that is invariant in the $\mathbb{R}$-direction)

Proof:

if Σ is convex, then let v be the transverse contact v.f.

set $H=-\alpha(v)$ (some contact form α for ξ)

cut off H near Σ (so it has compact support)

let v' be the contact v.f. associated (by lemma 1) to new function

flow of v' (which exist for all time since has compact support) gives ϕ

conversely, given ϕ let t be coordinate on $\mathbb{R}$

the vector field $v=\phi_*\frac{\partial}{\partial t}$ is a contact v.f. transverse to Σ

exercise: If Σ is a convex surface in (M,ξ), then show, using lemma above, that Σ has a neighborhood $\Sigma\times[-1,1]$ such that ξ is given by a 1-form

$$\alpha=\beta+u\,dt$$

β a 1-form on Σ and $u:\Sigma\to\mathbb{R}$

note no t dependence for β, u

<u>note</u>: with α as above

1) $\Sigma_\xi = \ker \beta$

2) for α to be contact we need

$$\alpha \wedge d\alpha = \beta \wedge (d\beta + du \wedge dt) + u\,dt \wedge \beta$$
$$= (\beta \wedge du + u\,d\beta) \wedge dt > 0$$

so $\boxed{\beta \wedge du + u\,d\beta > 0}$ 1

<u>lemma 3</u>:

let Σ be a surface in (M, ξ)
$\iota : \Sigma \to M$ the inclusion map
α a contact form for ξ
$\beta = \iota^* \alpha$

the surface Σ is convex
$\Longleftrightarrow$
$\exists$ a function $u : \Sigma \to \mathbb{R}$ s.t. $u\,d\beta + \beta \wedge du > 0$

<u>Proof</u>:

If Σ is convex we are done from above

If u exists then on $\Sigma \times \mathbb{R}$ consider the contact structure

$\ker(\beta + u\,dt)$

char fol^n on $\Sigma \times \{0\}$ and Σ are the same

$\therefore$ we have neighborhoods of $\Sigma \times \{0\}$ and Σ that are contactomorphic and contactomorphism sends $\frac{\partial}{\partial t}$ to a contact vector field transverse to Σ

dualize equation 1: fix an area form on Σ

so there is a vector field w on Σ such that

$$\iota_w \omega = \beta$$

note w is in $\ker\beta$ and so directs Σ_ξ

(i.e. tangent to Σ_ξ and 0 at singularities)

if Σ convex then

$$\begin{aligned}
&\beta\wedge du + u\,d\beta > 0\\
&\quad\|\\
&\beta\wedge du + u(\mathrm{div}_\omega w)\omega\\
&\quad\|\\
&\iota_w\omega\wedge du + u(\mathrm{div}_\omega w)\omega\\
&\quad\|\\
&(-du(w) + u\,\mathrm{div}_\omega w)\,\omega
\end{aligned}$$

recall $du\wedge\omega = 0$ so

$$\begin{aligned}
&\iota_w(du\wedge\omega) = 0\\
&\quad\|\\
&du(w)\,\omega - du\wedge\iota_w\omega\\
&\quad\|\\
&du(w)\,\omega + \iota_w\omega\wedge du
\end{aligned}$$

so $\boxed{-du(w) + u\,\mathrm{div}_\omega w > 0}$ 2

exercise: for a fixed β set of u satisfying 1 is convex

" " w " " 2 is convex

example (of non convex surface):

$\mathbb{R}^3$ coords (r,θ,z)

$M = \mathbb{R}^3/_{z\mapsto z+1}$

$\xi = \ker(dz + r^2 d\theta)$

$T_c = \{(r,\theta,z)\mid r=c\}$

characteristic fol^n on T_c is linear

note β above on T_c is $dz + c^2 d\theta$

so $d\beta = 0$

if $\omega = d\theta\wedge dz$ on T_c then $w = c^2\frac{\partial}{\partial z} - \frac{\partial}{\partial\theta}$

satisfies $\iota_w \omega = \beta$ so w directs char. fol$^{\underline{n}}$

and $\mathrm{div}_\omega w = 0$

$\therefore$ if T_c convex $\exists$ a function $u : T_c \to \mathbb{R}$

such that $-du(w) > 0$

so w decreases along flow lines

leaves of $(T_c)_\xi$ are closed $\cancel{\otimes}$

or dense $\cancel{\otimes}$

so T_c not convex

exercise:

let Σ be a surface in (M, ξ)

if one of the following is true then Σ is not convex

(1) Σ_ξ has a flow line from a negative to a positive singularity

(2) Σ_ξ has a dense leaf.

given a surface Σ

a singular foliation $\mathcal{F}$ on Σ

properly embedded arcs and simple closed curves

we say a multi-curve Γ divides $\mathcal{F}$ if

(1) $\Sigma \setminus \Gamma = \Sigma_+ \amalg \Sigma_-$

(2) Γ is transverse to $\mathcal{F}$ and

(3) there is a volume form ω on Σ and vector field w on Σ

such that (a) $\pm \mathrm{div}_\omega v > 0$ on $\Sigma_\pm$

(b) w directs $\mathcal{F}$

(c) w points out of Σ_+ along $\partial\Sigma_+ - (\partial\Sigma_+ \cap \partial\Sigma)$

exercise: if Γ_1, Γ_2 both divide $\mathcal{F}$ then Γ_1 and Γ_2 are isotopic through dividing curves

if Σ is a convex surface then near Σ we can write the contact form $\beta + u\,dt$ the multi-curve

$$\Gamma_\Sigma = \{x \in \Sigma : u(x) = 0\}$$

can assume 0 a regular value of u

is called the dividing set of Σ

Thm 4:

given a compact orientable surface Σ in (M, ξ) with $\partial\Sigma$ Legendrian Then

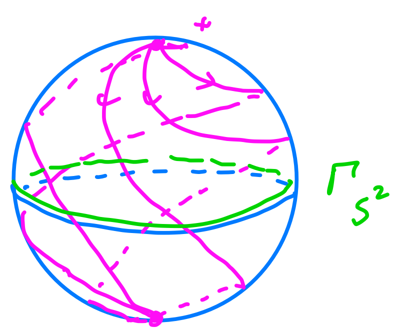
Σ is convex $\iff$ there is a dividing set for Σ_ξ

we will prove this theorem and the ones below later

but now we give a user's guide to convex surfaces and then see how they are used to study contact structures

examples:

1) S^2 unit sphere in $\mathbb{R}^3$ with $\xi = \ker(dz + r^2\, d\theta)$



indeed if $v = \frac{1}{2} r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z}$

then $\mathcal{L}_v \alpha = \alpha$ so v contact

and $\alpha(v) = z$ so $\Gamma_{S^2} = \{z = 0\}$

2) recall $T_c = \{(r,\theta,z) \mid r = c\} \subset \mathbb{R}^3/_{z \mapsto z+1}$

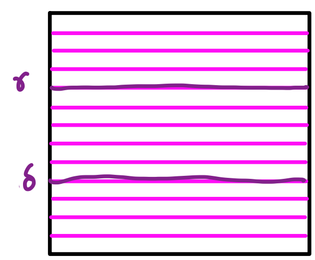
with $\xi = \ker(dz + r^2 d\theta)$

above we saw T_c <u>not</u> convex

$(T_c)_\xi$ is a linear foliation

choose c so slope is rational p/q

pick 2 orbits γ, δ of $(T_c)_\xi$

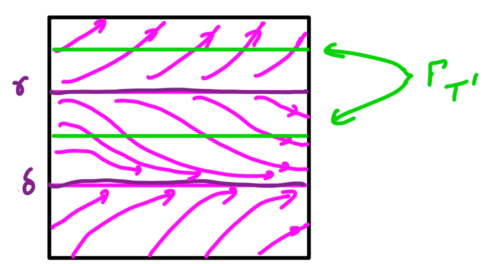


changed coordinates on torus so picture clearer

$T_c - (\gamma \cup \delta) = A_1 \cup A_2$ 2 annuli

push A_1 out a little and A_2 in a little to get T'

with T'_ξ



note the new torus has dividing curves so is convex

so a C^∞ small perturbation of the non-convex T_c is convex!

<u>note</u>: we could have perturbed T_c to have any even number of dividing curves

more generally we have

$Th^{\underline{m}}$ 5:

any closed surface is C^∞-close to a convex surface

if Σ contains Legendrian curves $L_1 \dots L_k$ with $tw_\xi(L_i, \Sigma) \leq 0$ for all i, then Σ may be C^0-isotoped near L_i and C^∞-isotoped away from the L_i to become convex

so convex surfaces are very common!

$Th^{\underline{m}}$ 6 (Giroux flexibility):

suppose
- Σ a compact surface in (M, ξ)
- Σ closed or has $\partial\Sigma$ Legendrian with non-positive twisting along each component of $\partial\Sigma$
- Σ is convex with dividing curves Γ_Σ and transverse contact vector field v
- $i : \Sigma \to M$ the inclusion map

let $\Gamma = i^{-1}(\Gamma_\Sigma)$ and $\mathcal{F}$ be <u>any</u> singular foliation on Σ that is divided by Γ

Then in any neighborhood U of Σ in M, there is an isotopy $\phi_s : \Sigma \to M$ for $s \in [0,1]$ such that

(1) $\phi_0 = i$

(2) ϕ_s is fixed on Γ

(3) $\phi_s(\Sigma) \subset U$ for all s

(4) $\phi_s(\Sigma)$ is transverse to v ($\therefore$ convex) with $\Gamma_{\phi_s(\Sigma)} = \Gamma_\Sigma$

(5) $(\phi_1(\Sigma))_\xi = \phi_1(\mathcal{F})$

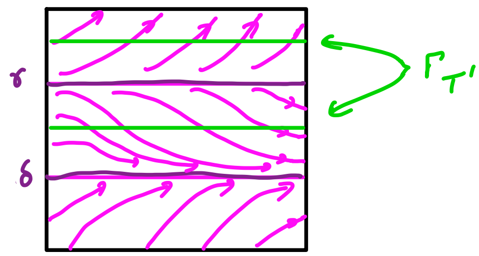
recall $Th^{\underline{m}}$ II.5 says Σ_ξ determines ξ near Σ, coupled with $th^{\underline{m}}$ above we see Γ_Σ more or less determines ξ near Σ

way easier to understand multi-curves than foliations!

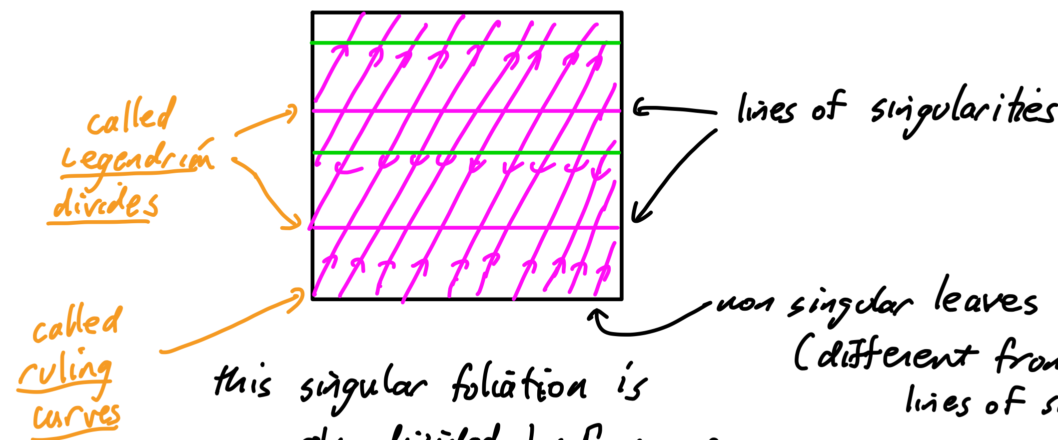
example:

this flexibility is very powerful!

in the example above we saw a torus with foliation



now consider the singular foliation



non singular leaves of slope s (different from slope of lines of singularities)

this singular foliation is also divided by Γ so we may isotop the torus to have this fol^n as characteristic fol^n!

this is surprising as fol^n is very non generic and we can realize any slope s ($\neq$ slope of sing. lines)

we call a torus with foliation as above a torus in standard form and it is determined by the slope r of the dividing curves and the slope s (any $s \neq r$) of the ruling curves

let Σ be a convex surface in (M, ξ)

Γ_Σ the dividing curves

a graph $G \subset \Sigma$ is called non-isolating if G is transverse to Γ_Σ

and every component of $\Sigma \setminus G$ intersects Γ_Σ

Th$^{\underline{m}}$ 7 (Legendrian Realization Principle or LeRP):

Σ a convex surface in (M,ξ)

G a graph in Σ that is non isolating

Then there is an isotopy of Σ (rel $\partial\Sigma$) through convex surfaces to Σ', s.t. G is contained in the characteristic fol$^{\underline{n}}$ of Σ'

a useful corollary is

Corollary 8:

If C is a simple closed curve in a convex surface Σ that nontrivially and transversely intersects Γ_Σ then Σ may be isotoped so that C is Legendrian on Σ

we can say a lot about Legendrian curves on a convex surface

Th$^{\underline{m}}$ 9:

let L be a Legendrian simple closed curve in a convex surface Σ that is transverse to, then

$$tw_\xi(L,\Sigma) = -\tfrac{1}{2}\#(L \cap \Gamma_\Sigma)$$

if $L = \partial\Sigma$, then this gives $tb(L)$, moreover

$$r(L) = \chi(\Sigma_+) - \chi(\Sigma_-)$$

we can also understand tightness using convex surfaces

Thm 10 (the Giroux Criterion):

Σ a convex surface in (M, ξ)

a vertically invariant neighborhood of Σ is tight

$\iff$

(1) $\Sigma = S^2$ and Γ_Σ is connected, or

(2) $\Sigma \neq S^2$ and Γ_Σ has no components bounding a disk

we end by seeing how to "transfer information" between convex surfaces

lemma 11:

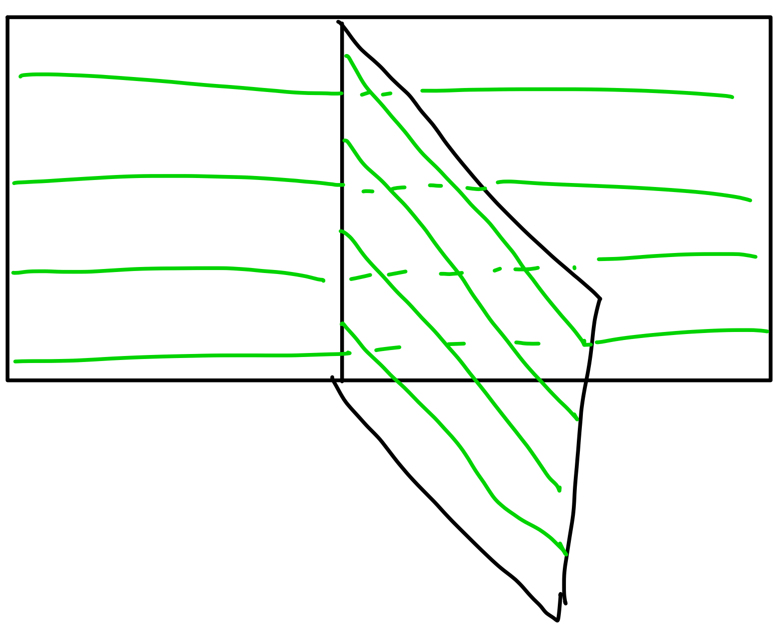
Σ, Σ' convex surfaces with dividing sets $\Gamma_\Sigma, \Gamma_{\Sigma'}$

$\partial\Sigma' \subset \Sigma$ a Legendrian curve

let $S = \Gamma_\Sigma \cap \partial\Sigma'$ and $S' = \Gamma_{\Sigma'} \cap \partial\Sigma'$

then between any two adjacent points of S there is one point of S', and vice-versa

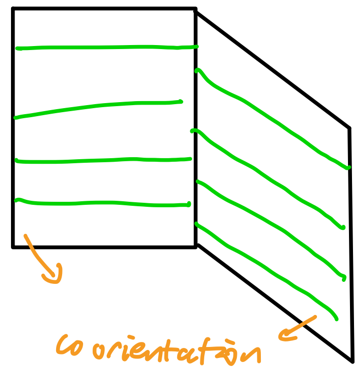
Pictorially



we can say a little more

lemma 12:

If Σ, Σ' are as in lemma 11 but $\partial\Sigma = \partial\Sigma'$ and look like



then one can "round the corner" to get a smooth convex surface with dividing set